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Math 112: Introductory Real Analysis

§ Lecture 5 (Feb 10, 2025)

Last time: Cardinality of sets

finite, countable, uncountable sets

Cantor's diagonal argument: $|S| < |2^S|$

$\mathbb{R}$ is uncountable.

Today: metric spaces

Def A set X, (whose elements we shall call points), is said to be a metric space if it is equipped with a metric (or a distance function), which is a function

$$d : X \times X \longrightarrow \mathbb{R}$$
$$(p, q) \longmapsto d(p, q)$$

"distance between p and q"

satisfying the following properties:

(positivity) $d(p,q) > 0$ if $p \neq q$, and $d(p,p) = 0$, for every $p, q \in X$

(symmetry) $d(p,q) = d(q,p)$, for every $p, q \in X$

(triangle inequality) $d(p,q) \leq d(p,r) + d(r,q)$, for every $p, q, r \in X$

Examples

- $\mathbb{R}$ with $d(x,y) := |x-y|$ is a metric space.
- More generally, Euclidean spaces $\mathbb{R}^k$ with
  $$d(\mathbf{x},\mathbf{y}) := |\mathbf{x}-\mathbf{y}| = \left(\sum_{i=1}^{k} (x_i-y_i)^2\right)^{\frac{1}{2}}$$
  is a metric space.

Note, every subset $Y \subseteq X$ of a metric space X is also a metric space.

Def An open interval (a,b) is $(a,b) := \{x \in \mathbb{R} \mid a < x < b\}$.
A closed interval $[a,b]$ is $[a,b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}$.

Def For $\mathbf{x} \in \mathbb{R}^k$ and $r > 0$, the open (resp. closed) ball B centered at $\mathbf{x}$ with radius r is defined to be the set
$$\{\mathbf{y} \in \mathbb{R}^k \mid |\mathbf{y}-\mathbf{x}| < r\} \quad (\text{resp. } \{\mathbf{y} \in \mathbb{R}^k \mid |\mathbf{y}-\mathbf{x}| \leq r\}).$$

<u>Def</u> Let X be a metric space.

For $p \in X$ and $r > 0$, the r-neighborhood of p (or an open ball with center p and radius r), $N_r(p)$, is

$$N_r(p) := \{q \in X \mid d(q,p) < r\}.$$

For $E \subseteq X$ and $p \in E$, we say p is an interior point of E if there is $r > 0$ such that $N_r(p) \subseteq E$.

write $\mathrm{int}(E)$ for the set of interior points

We say $E \subseteq X$ is open if every point of E is an interior point.

(In other words, if it is a union of open balls.)

We say $E \subseteq X$ is closed if its complement $E^c = X \setminus E$ is open.

(Rudin defines closed sets in terms of limit points; we'll see that these two definitions are equivalent.)

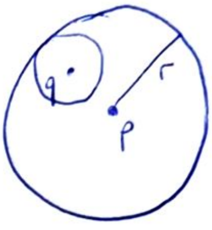
For $E \subseteq X$ and $p \in X$ (not necessarily $p \in E$), we say p is a limit point of E if $(N_r(p) \setminus \{p\}) \cap E$ is non-empty (in fact, infinite) for all $r > 0$.

write E' for the set of limit points

(We say $E \subseteq X$ is bounded if there is a (big enough) real number M and a point $q \in X$ such that $N_M(q) = X$.)

Thm Every open ball $N_r(p)$ is open.

proof) For any point $q \in N_r(p)$,



$$N_{r-d(p,q)}(q) \subseteq N_r(p), \text{ because}$$

$$d(q,s) < r - d(p,q) \Rightarrow d(p,s) \leq d(p,q) + d(q,s) < r \quad \blacksquare$$

Thm $E \subseteq X$ is closed iff every limit point of E is in E.

proof) Let $\overline{E} := \{p \in X \mid p \in E \text{ or } p \text{ is a limit point of } E\}$. $= E \cup E'$

"the closure of E"

$$= \{p \in X \mid N_r(p) \cap E \text{ is non-empty for all } r > 0\}.$$

$$= \{p \in X \mid N_r(p) \not\subseteq E^c \text{ for all } r > 0\}$$

$$= \left(\text{int}(E^c)\right)^c$$

Hence, if E is closed, then E^c is open, so $\overline{E} = \text{int}(E^c)^c = (E^c)^c = E$.

In the opposite direction, if $\overline{E} = E$, then $E = \text{int}(E^c)^c$,

so $E^c = \text{int}(E^c)$, meaning E^c is open (i.e. E is closed) $\blacksquare$

Note, $\overline{\overline{E}} = \overline{E}$ (and hence $\overline{E}$ is closed), and $\text{int}(E)$ is open.

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Thm

(a) Any union of open sets is open.

(Any intersection of closed sets is closed)

(b) Any finite intersection of open sets is open

(Any finite union of closed sets is closed)

proof)

(a) Suppose $\{G_\alpha\}$ is a collection of open sets.

Then, for any $p \in \bigcup_\alpha G_\alpha$, there is some α for which $p \in G_\alpha$

and some $r > 0$ such that $N_r(p) \subseteq G_\alpha$ (since G_α is open).

Then, $N_r(p) \subseteq \bigcup_\alpha G_\alpha$, and therefore $\bigcup_\alpha G_\alpha$ is open.

(b) Suppose $G_1, \cdots, G_n$ are open sets.

Then, for any $p \in \bigcap_{i=1}^{n} G_i$, there are $r_1, \cdots, r_n > 0$ such that

$N_{r_i}(p) \subseteq G_i$ for each i.

Let $r = \min\{r_1, \cdots, r_n\} > 0$.

Then, $N_r(p) \subseteq G_i$ for all $1 \leq i \leq n$,

and hence $N_r(p) \subseteq \bigcap_{i=1}^{n} G_i$, and therefore $\bigcap_{i=1}^{n} G_i$ is open

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<u>Rmk</u> Let X be a set.

Any collection of subsets $\{G_\alpha\}_{\alpha\in I}$ of X which is closed under

(arbitrary) unions

and finite intersections

is called a topology on X.

So, when X is a metric space, the collection of all subsets of X that can be expressed as a union of open balls (i.e. the collection of all open sets) is a topology on X.

<u>Rmk</u> If $\{G_\alpha\}_{\alpha\in I}$ is a topology on X,

then for any subset $Y \subseteq X$, there is an induced topology on Y

given by $\{G_\alpha \cap Y\}_{\alpha\in I}$.